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KAPPA FUNCTION AS A UNIFYING FRAMEWORK FOR DISCRETE POPULATION MODELING

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ABSTRACT. This paper develops a unified way to describe the various generalized discrete-time nonlinear dynamical models with density dependence, Allee effects, and parasitoids. We show how the kappa function can be used to describe the probabilities involved in intra- or interspecific encounters, namely, (i) the probability of surviving to the next generation in the absence of parasitoids or Allee effects, (ii) the encounter probability associated with Allee effects, and (iii) the probability of escaping parasitism in the presence of parasitoids. Having introduced a phenomenological framework of modeling via the kappa function, we then provide a realistic mechanism through stochastic encounters, responsible for generating the kappa function to any of the three involved probabilities. The unified modeling through the kappa function yields insights into how abundances influence species interactions. It is now straightforward to use this unified modeling to analyze and investigate its consequences in species dynamics.

KEY WORDS: Population dynamics, kappa distribution, Allee effect, host-parasitoid models, discrete-time models.

1. Introduction. Discrete-time models of ecological populations with their rich dynamical structure and complexity have been of considerable mathematical and ecological interest for over a half a century. For biological populations with

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nonoverlapping generations, population growth takes place in discrete time steps and is commonly described by difference equations (Elaydi [2005, 2008]). Moreover, discrete-time population models have proved highly effective for analyzing time series data arising from experimental and observational studies (Dennis and Taper [1994], Ives et al. [2003], Costantino et al. [2005]).

This paper proposes a unified framework for modeling populations via difference equations. This includes modeling density dependence, parasitism, and Allee effects, and focuses on the probability mechanisms involved. Unified approaches to population modeling have been attempted by many authors (May [1978], May et al. [1981], Royama [1992], Mills and Getz [1996], Brännström and Sumpter [2005]). Such approaches can lead to deeper insights into modeling, the relationships among various models, how individual actions affect population dynamics, and theoretical developments about species abundance distributions (McGill et al. [2007]). In one generalized approach to single-species modeling, Brännström and Sumpter [2005] were able to derive many of the models in the ecology literature by introducing spatial clustering of individuals in the form of a negative binomial distribution. Following this approach, single-species models are represented by the difference equation

$$H_{t+1} = H_t \cdot g(H_t),\ g(H_t) = u(H_t) \cdot r(H_t) \cdot I(H_t), \tag{1a}$$

where H_t is the abundance of the population at generation or time period t, and $g(H_t)$ is the net individual fitness. (The abundance represents either the size or density, and hereafter will be called size.) Here the fitness is modeled as the product of three functions, the probability of surviving to the next generation when affected only by intraspecific competition, represented by a monotonically decreasing function $u(H_t)$, the rate of reproduction, $r(H_t)$, that is the number of offspring that survives to the next generation, and the probability of avoiding reproduction failure due to Allee effects represented by a monotonically increasing function $I(H_t)$. In the presence of a parasitoid, the standard modeling of a host-parasitoid system involves

$$H_{t+1} = H_t \cdot g(H_t) \cdot p(P_t),\ P_{t+1} = H_t \cdot \varsigma(H_t) \cdot [1 - p(P_t)], \tag{1b}$$

where $p(P_t)$ is the probability of the host to escape from parasitism; $\varsigma(H_t)$ is the parasitoid multiplicity, i.e., the eggs laid by a parasitoid on a single host that survive. Our focus will be on the probability functions $u(H_t)$, $I(H_t)$, and $p(P_t)$, and we will here assume that $r(H_t)$ and $\varsigma(H_t)$ are constant. Note that the probability of host survival in the absence of the parasitoid does not depend on the parasitoid size, while the probability of the host of escaping parasitism does not depend on the host size.

The forms of $u(x)$ and $p(x)$ are commonly given by an exponential or a rational function. These functions were generalized by many authors including Hassel et al. [1976] and Brännström and Sumpter [2005]. Most of these models may fall under the framework of the mono-parametrical kappa function (Livadiotis and McComas [2009, 2013], Livadiotis [2015a]),

$$f(x\,;\,\kappa) = \left(1 + \frac{1}{\kappa}\cdot x\right)^{-\kappa-1} \kappa \in (0,\infty). \tag{2}$$

Note that x may represent the size N of a population or a function $\Phi(N)$ of the size. The governing parameter κ is an inverse measure of the degree of aggregation or clustering in the items being encountered. When κ tends to zero, the probability (2) approaches the functional form $1/(1+x)$ of the Beverton-Holt's model [1957] characterized by extreme aggregation; as will be explained later, the variable x can be a physically meaningful function of κ, e.g., $x \rightarrow x \cdot \kappa/(\kappa+1)$. On the other end, when κ tends to infinity, we have the exponential form e^{-x} of the Ricker's model [1957] that has no aggregation.

For multi-species models, it was May [1978, 1981], who initiated the use of kappa function to modify the Nicholson–Bailey host-parasitoid model. If P is the size of the parasite, then the probability of the host escaping parasitism is given by (2) with $x = \gamma \cdot P$, where γ is the attack rate of the parasitoid, and κ is the aggregate parameter of the parasitoid. In addition to parasitism, and individual fitness, the framework of kappa function may include the Allee effect phenomenon as was pointed out by Johansson and Sumpter [2003]. This may be done by requiring, for example, that two or more individuals are needed per resource site for offspring to be produced.

The above formulations lie in the theoretical framework of the kappa function used for describing the particle energy distribution of various systems with inter-particle interactions, such as, the space plasmas (e.g., Livadiotis [2015a], and references therein). The use of the kappa function has become increasingly widespread across physics and other disciplines, with the number of papers following, remarkably, an exponential growth (Livadiotis [2015b]). The kappa function was empirically derived half a century ago as a suitable fitting-model describing observations of space and other plasma particle populations (Olbert [1968], Vasyliunas [1968]). A breakthrough in the field came with the connection of the kappa function with the statistical framework of nonextensive statistical mechanics (Tsallis [2009]), that is, a generalization of the classical statistical mechanics of Boltzmann-Gibbs (Livadiotis and McComas [2009]). There was an outburst of publications studying and applying the kappa function and its associated statistical mechanics in a variety of disciplines. Few examples are the following: in sociology-sociometry, e.g., the internet (Abe and Suzuki [2003]), in citation networks of scientific papers (Tsallis and De Albuquerque [2000]), urban agglomeration (Malacarne et al. [2001]); in linguistics (Montemurro [2001]); in economics (Borland [2002]); in biochemistry (Andricioaei and Straub [1996]); in applied statistics (Habeck et al. [2005]); in nonlinear dynamics (Borges et al. [2002]); in physical chemistry (Livadiotis [2009]); space physics (Livadiotis et al. [2012]).

The purpose of this paper is to develop a general modeling and formulations in multi-species competition models with Allee effects and parasitoids using the framework of kappa function. This can be used to derive various probabilities related to this modeling, namely, the probabilities (i) of surviving to the next generation in the absence of parasitoids or Allee effects, (ii) of avoiding reproduction failure due to Allee effects, and (iii) of escaping parasitism in the presence of parasitoids.

2. Materials and methods.

2.1. Modeling framework via the kappa function. Let the kappa function of some variable x (2), which represents the size of species N, or it may be given by some simple function of N, $x = \Phi(N)$ (and of other parameters characteristic of the model). The governing parameter of the kappa function is the kappa index κ. Different values of κ correspond to different "states" of the system, ranking from the state called "Equilibrium" when $\kappa \to \infty$, where the function is reduced to the exponential, to the furthest state from Equilibrium, the Anti-Equilibrium, when $\kappa \to 0$, where the kappa function has a rational or power-law asymptotic behavior. The Equilibrium/Anti-Equilibrium terms are borrowed from the usage of kappa function in statistical mechanics (e.g., Livadiotis [2015a]).

Here we examine the probabilities of the host to survive among its own species individuals, to escape parasitism, and to avoid Allee effects, and connect them with the kappa function (Figure 1).

Simple population models of one-species (in the absence of parasitoids or Allee effects) have similar behavior. Let the standard modeling $H_{t+1}/H_t = u(H_t)$. Now the involved probability is $u(H_t)$, that is the probability of an individual to survive to the next generation. The two basic models of the survival probability $u(x)$ are those of Ricker [1954] and Beverton–Holt [1957], respectively, given by the functions,

$$u(x) = e^{-x}, \text{and } u(x) = 1/(1+x)\,, \text{ with} \, x \equiv c(R) \cdot (H_t/K - 1), \tag{3}$$

where R is a characteristic rate per generation (intrinsic growth rate for the Ricker model and proliferation rate for the Beverton–Holt model); c is constant related to the characteristic rate, where $c(R) = R$ for the Ricker model and $c(R) =$ 1-R^{-1} for the Beverton–Holt model; K is the carrying capacity. Both models are limiting cases of the Hassell et al. model [1976], and they can be written in the form

$$u(x) = \left(1 + \frac{1}{\kappa} \cdot x\right)^{-\kappa-1}, \text{ with} \, x \equiv c_\kappa(R) \cdot \left(\frac{H_t}{K} - 1\right), \; c_\kappa(R) \equiv \frac{R \cdot \kappa}{\kappa + R^2 (R-1)^{-1}}. \tag{4}$$

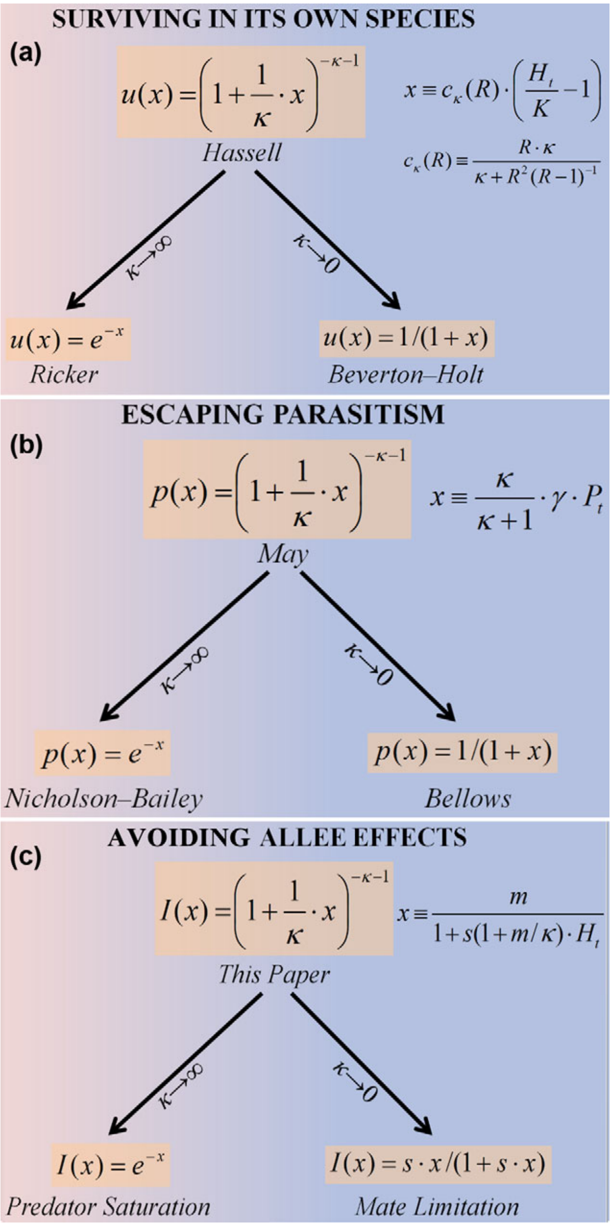

FIGURE 1. The probabilities of the host species of (a) surviving, (b) escaping parasitism, and (c) avoiding Allee effects, and their connection with the framework of the kappa function.

The Ricker model is obtained when $\kappa \to \infty$ and the Beverton–Holt model when $\kappa \to 0$. The model is developed so that K to be the carrying capacity for all the values of κ.

In the general host-parasitoid system, the probability that a host escapes from parasitism can be described by an exponential (Nicholson and Bailey [1935]) and a rational (Bellows [1981]) models,

$$p(x) = e^{-x}, \text{ and } p(x) = 1/(1+x) \text{ with } x \equiv \gamma P_t, \tag{5}$$

where γ is the searching efficiency of the parasitoid. These models constitute the two extreme cases of May's general model

$$p(x) = \left(1 + \frac{1}{\kappa} \cdot x\right)^{-\kappa-1}, \text{ with } x \equiv \frac{\kappa}{\kappa+1} \cdot \gamma \cdot P_t. \tag{6}$$

The Nicholson and Bailey model is obtained for $\kappa \to \infty$ and the Bellows model for $\kappa \to 0$ (Livadiotis et al. [2015]). (The exponent $-(\kappa+1)$, instead of $-\kappa$, is preferred in order to be aligned with the standard kappa function, e.g., Livadiotis [2015a]).

Next we use the kappa function to describe in a unified way the primary models of the Allee effect. A standard method of modeling species suffering from strong Allee effect is via the equation

$$u(H_t) = u_0(H_t) \cdot I(H_t), \tag{7}$$

where the function $u_0(H)$ is the host fitness in the absence of an Allee effect; $I(H)$ represents the Allee effect on the fitness of the population (e.g., Dennis [1989], Scheuring [1999], Livadiotis and Elaydi [2012]) and it is given by the probabilities of (a) escaping predation by a predator with a saturating functional response, or (b) finding a mate:

$$(a)\, I(x) = \exp\left(-\frac{m}{1 + s \cdot x}\right), \; (b)\, I(x) = \frac{s \cdot x}{1 + s \cdot x}, \text{ with } x \equiv H_t, \tag{8}$$

where m represents the predation intensity; the interpretation of s depends on the model, i.e., for (8a) s is proportional to the "handling time," which is the duration of the capture and eating phase, and for (8b) s is the individual's searching efficiency.

Both the above primary Allee effect models (8) are limiting cases of the following kappa function

$$I(x) = \left(1 + \frac{1}{\kappa} \cdot x\right)^{-\kappa-1}, \text{ with } x = \frac{m}{1 + s(1 + m/\kappa) \cdot H_t}\, m > 0. \tag{9}$$

This reduces to the predator saturation model (8a) for $\kappa \to \infty$ and the mate limitation model (8b) for $\kappa \to 0$. The scheme in Figure 1 shows the unifying role of the kappa function in all three probabilities described above.

2.2. Origin of the kappa function in population dynamics. Having introduced a phenomenological framework for modeling the probabilities involved in the fitness, we now may ask if there exists any realistic mechanism that lies on the previous abstract analysis. Indeed, we show that stochastic encounters can act as a mechanism responsible for generating a kappa function to any of the three probabilities of Figure 1.

In dynamical models of density-dependent competing populations, host-parasite systems, and populations experiencing Allee effects, there are often terms in the mathematical expressions that represent probabilities or frequencies of encounters. Some of the more popular terms arise from formulating encounters as a simple discrete stochastic birth process.

Suppose $Y(t)$ is the number of encounters that an animal has had with the objects of search or avoidance (food items, prey, co-members of its own species, predators), starting from an initial time $t = 0$. In some versions, t is not necessarily time but could be the searched area. Additions to the count $Y(t)$ in a small interval Δt are governed by a function, λ_y, commonly called the "birth rate" although in the present context encounters are not being born as such. Specifically, the probability of an additional encounter in Δt, given the value of $Y(t)$ is y, is assumed to be

$$\mathrm{P}\left[Y(t + \Delta t) = y + 1 \mid Y(t) = y\right] = \lambda_y \Delta t. \tag{10}$$

With the additional assumption that the probability of two or more encounters in Δt is negligible, equation (10) leads to a system of differential equations for the probabilities $p_y(t) = \mathrm{P}[Y(t) = y]$, $y = 0, 1, \ldots$; (these differential equations are listed in most books on stochastic processes, e.g., Bailey [1964]). With the initial condition that $Y(0) = 0$ (or $p_0(0) = 1$), the equations can be solved recursively to yield

$$p_y(t) = e^{-\lambda_y t} \lambda_{y-1} \int_0^t e^{\lambda_y \tau} p_{y-1}(\tau)\, d\tau, \tag{11a}$$

starting from

$$p_0(t) = e^{-\lambda_0 t}. \tag{11b}$$

Different forms for the birth rate function λ_y correspond to different temporal or spatial patterns in encounters (e.g., Abbott [2011]). For instance, a simple model of temporal or spatial aggregation takes the birth rate to be

$$\lambda_y = \lambda_0 + \gamma \cdot y, \tag{12a}$$

where λ_0 and γ are positive quantities; the more encounters an animal has had, the more likely it is the animal will have another encounter.

A main point is that the zero probability (11b) has the same exponential decay form for any birth rate function, the dependence being only on the birth rate λ_0 at $y = 0$ encounters. The zero probability is the probability of no encounters in time t and plays an important role when encounter avoidance or failure has dynamical consequences. Survival (avoiding predation or cannibalism), mating (finding conspecifics), and feeding (finding food, prey, or hosts) are potential processes modeled with (11b). In particular, λ_0 is a sort of density, or average number of first encounters over one unit of time or space. We expect λ_0 to be roughly proportional to N, the average density of the objects being encountered,

$$\lambda_0 = \beta\, N \, or \, \lambda_0 = \beta\, \Phi(N), \tag{12b}$$

where $\Phi(N)$ is some monotonic function of the size. For example, for the three probabilities in Figure 1 we have $\Phi(N) = (A + B \cdot N)^{\pm 1}$, where A and B are some constants depending on the model parameters; exponent $+1$ corresponds to the probabilities of surviving in its own species (Figure 1a) and of escaping parasitism (Figure 1b), while exponent -1 corresponds to the probability of avoiding Allee effects (Figure 1c).

In terms of N, the probability of zero encounters in a unit of time/space becomes

$$\mathrm{P(zero\ encounters)} = e^{-\beta N}. \tag{13a}$$

The complement probability gives the chance of one or more encounters:

$$\mathrm{P(non - zero\ encounters)} = 1 - e^{-\beta N}. \tag{13b}$$

In population dynamics, the exponential function in (13a,b) is used as encounter terms in the Thompson [1929] and Nicholson and Bailey [1935] host-parasitoid models, the Ricker [1954] model of stock-recruitment, and the Larvae-Pupae-Adults model of Dennis et al. [1995], among others.

Search (or avoidance) is likely to be a heterogeneous process. Detection conditions, detection abilities, and mobilities or escape abilities will vary from day to day and place to place. Heterogeneity can be modeled by allowing the detection parameter β, called "searchability," to have a continuous probability distribution of its own on the positive real line. In this formulation, (13a) represents a conditional probability, for a given value of β (after it has been drawn from its probability distribution). If the probability density function of β is denoted by $\phi(\beta)$, then the unconditional probability of zero encounters is given by the Laplace transform of $\phi(\beta)$,

$$\text{P}(\text{zero encounters}) = \int_0^\infty e^{-\beta\,\Phi(N)}\phi(\beta)\,d\beta. \tag{14}$$

The searchability distribution function, $\phi(\beta)$, can be identified by the Gamma distribution with shape κ and scale given by the mean $\langle\beta\rangle$, that is, the reduced chi-square distribution for noninteger d-degrees of freedom and average searchability $\langle\beta\rangle$,

$$D(\beta)d\beta = \frac{(\frac{1}{2}d)^{\frac{1}{2}d}}{\Gamma(\frac{1}{2}d)} \cdot (\beta/\langle\beta\rangle)^{\frac{1}{2}d-1} \cdot \exp\left(-\tfrac{1}{2}d\cdot\beta\right)\ d(\beta/\langle\beta\rangle). \tag{15}$$

Then, the integration in equation (14) gives the kappa function for $\kappa \equiv \frac{1}{2}d$ and $x \equiv \langle\beta\rangle \cdot \Phi(N)$,

$$p(x) = \left(1 + \frac{1}{\kappa}\cdot x\right)^{-\kappa-1}. \tag{16}$$

The expression $p(x)$ (the kappa function) here arises as the zero probability resulting from "mixing" a gamma distribution with the zero probability of a birth process encounter distribution such as a Poisson distribution in order to account for heterogeneity of encounters among individuals. The above stochastic mechanism of heterogeneity responsible for generating the kappa function has been used in statistical physics (under the term "superstatistics"), with β having the role of the inverse temperature (e.g., Beck [2001], Beck and Cohen [2003], Tsallis [2009]; the mechanism has been also applied to space physics in Schwadron et al. [2010]; evidence of the connection of $D(\beta)$ with gamma distribution can be found in Livadiotis [2015b]). The mechanism of mixing a gamma distribution with a Poisson distribution to get a negative binomial distribution has been known in statistics for almost a century (Greenwood and Yule [1920]) and has been well known in ecology for much of that time (e.g., Fisher et al. [1943], Pielou [1969], Chave [2004], Martin et al. [2005]).

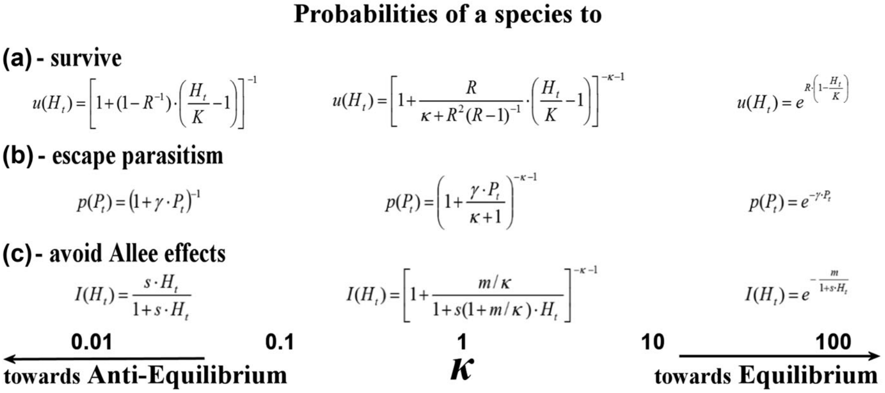

FIGURE 2. The probabilities of a species (a) to survive among its own species individuals, $u(H_t)$, (b) to escape parasitism, $p(H_t)$, and (c) to avoid reproduction failure due to Allee effects, $I(H_t)$, were considered and generalized in this paper using the kappa function. (a) As the kappa index decreases approaching $\kappa \to 0$ (noted as Anti-Equilibrium), then the survival probability reduces to Beverton–Holt model, while as the kappa index increases approaching $\kappa \to \infty$ (noted as Equilibrium), the survival probability reduces to the Ricker model; (b) the probability of avoiding Allee effects reduces correspondingly to "predator saturation" when $\kappa \to 0$ and "mate limitation" when $\kappa \to \infty$; (c) finally, the probability of escaping parasitism reduces correspondingly to the Bellows model when $\kappa \to 0$, and Nicholson–Bailey model when $\kappa \to \infty$.

3. Results. We provide the final modeling of population dynamics, where the kappa function is involved in the formalism of the probabilities of the host to survive, to escape parasitism, and to avoid Allee effects (Figure 2).

(i) Model to describe the dynamics of one species:

– without Allee effects

$$H_{t+1} = r \cdot H_t \cdot \left[1 + \frac{R}{\kappa + R^2(R-1)^{-1}} \cdot \left(\frac{H_t}{K} - 1\right)\right]^{-\kappa-1} \tag{17a}$$

– with Allee effects

$$\begin{aligned} H_{t+1} = r \cdot H_t \cdot &\left[1 + \frac{R}{\kappa + R^2(R-1)^{-1}} \cdot \left(\frac{H_t}{K} - 1\right)\right]^{-\kappa-1} \\ &\cdot \left[1 + \frac{m/\kappa}{1 + s(1+m/\kappa) \cdot H_t}\right]^{-\kappa-1} \end{aligned} \tag{17b}$$

– with Allee effects and a parasitoid

$$\begin{aligned} H_{t+1} &= r \cdot H_t \cdot \left[1 + \frac{R}{\kappa + R^2 (R-1)^{-1}} \cdot \left(\frac{H_t}{K} - 1\right)\right]^{-\kappa-1} \\ &\quad \cdot \left[1 + \frac{m/\kappa}{1 + s(1 + m/\kappa) \cdot H_t}\right]^{-\kappa-1} \\ &\quad \cdot \left(1 + \frac{\gamma \cdot P_t}{\kappa + 1}\right)^{-\kappa-1} \\ P_{t+1} &= \varsigma \cdot H_t \cdot \left[1 - \left(1 + \frac{\gamma \cdot P_t}{\kappa+1}\right)^{-\kappa-1}\right]. \end{aligned} \tag{17c}$$

It is likely that the kappa indices are different for the three probabilities; then, we have the model,

$$\begin{aligned} H_{t+1} &= r \cdot H_t \cdot \left[1 + \frac{R}{\kappa_1 + R^2 (R-1)^{-1}} \cdot \left(\frac{H_t}{K} - 1\right)\right]^{-\kappa_1-1} \\ &\quad \cdot \left[1 + \frac{m/\kappa_2}{1 + s(1 + m/\kappa_2) \cdot H_t}\right]^{-\kappa_2-1} \\ &\quad \cdot \left(1 + \frac{\gamma \cdot P_t}{\kappa_3 + 1}\right)^{-\kappa_3-1} \\ P_{t+1} &= \varsigma \cdot H_t \cdot \left[1 - \left(1 + \frac{\gamma \cdot P_t}{\kappa_3+1}\right)^{-\kappa_3-1}\right]. \end{aligned} \tag{17d}$$

Notes: (i) The parameter K coincides exactly to the carrying capacity only when $r = 1$ and in the absence of Allee effects ($m = 0$ or $s = 0$). (ii) Certainly, not all models are described by the modeling formulations presented in this paper. An example is the one-species theta Ricker model $u(x) = \exp\{r[1 - (x/K)^{\theta}]\}$ (Berryman [1999]).

4. Discussion. This paper presented a unified approach to modeling the population dynamics of biological species, developing a general formulation of the discrete-time nonlinear models with Allee effects and parasitoids. One significant effort of this theoretical analysis is the usage of the kappa function to model the probabilities involved, namely, (i) of surviving to the next generation in the absence of parasitoids or Allee effects, (ii) of avoiding reproduction failure due to Allee effects, and (iii) of escaping parasitism in the presence of parasitoids. The two basic models of each of these probabilities are those described by the exponential and rational function. The kappa function deforms these two extremes in a whole

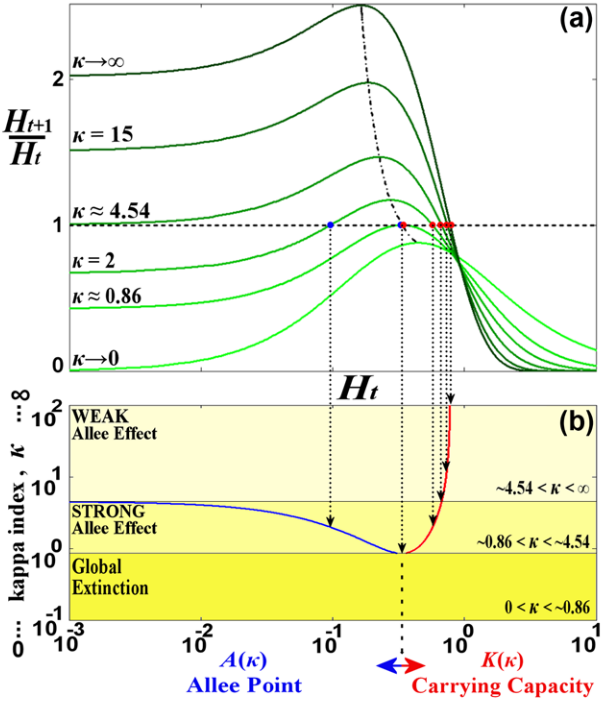

FIGURE 3. Variation of the kappa index and its influence on the Allee effect. (a) The fitness H_{t+1}/H_t is depicted in terms of H_t, and includes the probabilities of surviving in its own species and of avoiding Allee effects, in the absence of a parasitoid. The graph illustrates the transition of weak Allee effect, to strong Allee effect, and finally to global extinction (on a semi-log scale). (b) The graph shows the dependence of the carrying capacity (red) and Allee point (blue) as a function of the kappa index (on a log-log scale). (Notes: $K = 1$, $\gamma = 0$, $r = 1$, $R = 2.7$, $m = 2$, $s = 3$, and $\kappa_1 = \kappa_2$).

"spectrum" of different models, depending on the value of the governing parameter, that is, the kappa index.

In Figure 3, the fitness H_{t+1}/H_t is depicted in terms of H_t, and includes the probabilities of surviving and avoiding Allee effects, as given by equation (17b), in the absence of a parasitoid ($\gamma = 0$). For the illustrated example, the reproduction rate is ignored ($r = 1$), and the carrying capacity in the absence of Allee effect is taken fixed to $K = 1$; the rest of the parameters are $R = 2.7$, $m = 2$, $s = 3$, and the plots are shown for various values of the kappa index κ ($\kappa_1 = \kappa_2$), spanning the whole interval from $\kappa \to 0$ to $\kappa \to \infty$.

We observe that a variation of the kappa index causes a transition from weak to strong Allee effect, and finally, to global extinction. In particular, we find that the carrying capacity decreases as the kappa index increases. On the other hand, it is well known that the carrying capacity is an inverse measure of the strength of the Allee effect. Therefore, the Allee effect becomes stronger as the kappa index decreases. The transition from the weak to the strong Allee effect occurs for a certain kappa index that depends on the values of the other parameters, that is, $\kappa \approx 4.54$ for the illustrated example. For even smaller kappa indices, the Allee point (threshold of strong Allee effect) and the carrying capacity are getting closer and become equal when $\kappa \approx 0.86$. This limiting value of kappa index, which also depends on the values of the parameters, indicates the strongest Allee effect that can be described through the model (17b). Even smaller kappa indices lead to a global extinction of the species. This method imposes a powerful optimization helpful in species dynamics modeling. Future models should take into account the mathematical restrictions imposed by the kappa function (Livadiotis and McComas [2010]), in order to obtain physically meaningful species systems.

The modeling of population dynamics under the unified scheme of the kappa function (Figure 3) creates new connections between the species dynamics and the discipline of statistical mechanics, and specifically, to the framework of nonextensive statistical mechanics. It is now straightforward to use the unified modeling through the kappa function and analyze and investigate its consequences in species dynamics. Finally, the presented approach can lead to deeper insights into the connection of the kappa function with the species abundance distributions and their statistical mechanics.

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